Over- and Underrelaxation for Linear Systems with Weakly Cyclic Jacobi Matrices of Index p

P. Wild and W. Niethammer
Institut für Praktische Mathematik
Universität Karlsruhe
D-7500 Karlsruhe, West Germany

Submitted by R. S. Varga

ABSTRACT

D. Young's results from 1954 concerning the application of the successive-overrelaxation (SOR) method to linear systems $Ax = \mathbf{b}$ with matrices possessing "property A" were generalized by R. S. Varga in 1959 to systems with Jacobi matrices J_A which are weakly cyclic of index p; if the eigenvalues of J_A^p are nonnegative, then for the optimal parameter ω_{opt} there holds $1 \leqslant \omega_{\text{opt}} < 1 + 1/(p-1)$. Here a different proof of Varga's results is given and it is shown that $1 - 1/(p-1) < \omega_{\text{opt}} \leqslant 1$ holds if the eigenvalues of J_A^p are nonpositive. Further exact intervals for those parameters ω which yield convergence of the SOR method are given in terms of Chebyshev polynomials.

1. INTRODUCTION

Given a nonsingular system of linear equations

$$A\mathbf{x} = D\mathbf{x} - L\mathbf{x} - U\mathbf{x} = \mathbf{b}. \tag{1.1}$$

D a block diagonal matrix, L and U strictly lower and upper triangular matrices respectively, we consider the successive-overrelaxation (SOR) method

$$D\mathbf{x}^{(m)} = D\mathbf{x}^{(m-1)} + \omega\left(L\mathbf{x}^{(m)} - D\mathbf{x}^{(m-1)} + U\mathbf{x}^{(m-1)} + \mathbf{b}\right),$$
$$m = 1,2,\ldots, \qquad 0 \neq \omega \in \mathbb{R}. \tag{1.2}$$

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This iterative method is described in nearly all textbooks on numerical analysis; we use the notation of Varga [17].

Let $\mathbf{x}$ denote the exact solution of (1.1), and let $\mathbf{e}^{(m)} := \mathbf{x}^{(m)} - \mathbf{x}$ be the error at the mth step of (1.2); then

$$\mathbf{e}^{(m)} = L_\omega \mathbf{e}^{(m-1)}, \qquad \text{where} \quad L_\omega := (D - \omega L)^{-1}[(1-\omega)D + \omega U]. \quad (1.3)$$

It is well known that (1.2) is convergent, for arbitrary $\mathbf{x}^{(0)}$, to the solution $\mathbf{x}$ iff the spectral radius of L_ω, $\varrho(L_\omega)$, satisfies $\varrho(L_\omega) < 1$. This number $\varrho(L_\omega)$ yields a measure for the asymptotic decrease of $\|e^{(m)}\|$ (cf. Varga [17, p. 67]); thus, let us call $\varrho(L_\omega)$ the *asymptotic convergence factor* of the SOR method (1.2).

For arbitrary systems (1.1), very little is known in general about the optimal relaxation parameter ω_{opt} (which minimizes the asymptotic convergence factor as a function of ω). For the special but important class of matrices with "property A" D. Young found in 1954 his famous result on ω_{opt} [19], which was generalized by Varga in 1959 [16; 17, Chapter 4]. One considers the block Jacobi matrix $J_A := D^{-1}(L + U)$ associated with (1.1).

DEFINITION 1 (Varga [17, p. 39]). J_A is said to be *weakly cyclic of index p* if there exists a permutation matrix P such that PJ_AP^T has the form

$$PJ_AP^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & B_1 \\ B_2 & 0 & \cdots & 0 & 0 \\ 0 & B_3 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & B_p & 0 \end{bmatrix}, \quad (1.4)$$

where the null diagonal submatrices are square.

In the notation of Varga [17, p. 101], the matrix PJ_AP^T of (1.4) is said to be *consistently ordered*. In the following, we assume that J_A is always given in the normal form of the right-hand side of (1.4). For such matrices J_A, Varga derived the essential relation

$$(\lambda + \omega - 1)^p = \lambda^{p-1}\omega^p\mu^p \quad (1.5)$$

between the eigenvalues λ of L_ω and μ of J_A. Under the assumption that the eigenvalues μ^p of J_A^p satisfy $0 \leqslant \mu^p \leqslant \varrho(J_A^p) < 1$, Varga showed that ω_{opt} is the unique positive solution of the equation

$$[\varrho(J_A)\omega]^p = p^p(p-1)^{1-p}(\omega - 1); \tag{1.6}$$

for ω_{opt}, there holds

$$1 \leqslant \omega_{\text{opt}} < \frac{p}{p-1} = 1 + \frac{1}{p-1}, \tag{1.7}$$

i.e., overrelaxation yields the minimal asymptotic convergence factor.

If J_A is weakly two-cyclic and skew-symmetric, then $\omega_{\text{opt}} < 1$ (Niethammer [9]), i.e., underrelaxation yields the optimal acceleration of convergence. A weakly three-cyclic matrix appears in connection with a linear least-squares problem; here again underrelaxation is useful (see Chen [1], Plemmons [12], Niethammer, de Pillis, and Varga [11]). In both examples the eigenvalues of J_A^p are nonpositive; for ω_{opt} there holds

$$1 - \frac{1}{p-1} = \frac{p-2}{p-1} < \omega_{\text{opt}} \leqslant 1. \tag{1.8}$$

We shall give a different proof of Varga's results and derive exact convergence intervals, i.e., we determine intervals $I \subset (0,2)$ such that the SOR method converges for $\omega \in I$ and diverges for $\omega \notin I$. Results in this direction have also been obtained by Hadjidimos, Li, and Varga [6], who applied the Schur-Cohn theorem to decide whether all solutions λ of (1.5) are inside the unit disk or not. Incidentally, the geometric derivation presented here gives insight into when over- and underrelaxation are appropriate respectively.

Following an idea of Gutknecht (see Gutknecht, Niethammer, and Varga [5]), for a given linear system with a weakly p-cyclic Jacobi matrix J_A, we introduce the iterative method

$$\mathbf{x}^{(m)} = \omega J_A \mathbf{x}^{(m-1)} + (1-\omega)\mathbf{x}^{(m-p)} + \omega \mathbf{c}, \qquad m \geqslant p, \tag{1.9}$$

which we will call *p-step relaxation*. It is shown in Theorem 1 that (1.9) converges iff the SOR method converges; in the case of convergence, the SOR method is p times faster than the p-step relaxation.

Thus, for deriving results on the SOR method we can restrict ourselves to the easier study of the p-step relaxation. Here, the mapping

$$q_\omega(\lambda) := \frac{1-(1-\omega)\lambda^p}{\omega\lambda} \tag{1.10}$$

plays an important role. Let $U(\omega)$ denote the complement of the image of the closed unit disk under q_ω, and let $U_\eta(\omega)$ denote the complement of the image of the open disk with radius η. Then Theorem 2 says that the p-step relaxation (and SOR) converges for some $0 \neq \omega \in \mathbb{R}$ iff the spectrum $\sigma(J_A)$ is contained in $U(\omega)$; if $\sigma(J_A) \subset U_\eta(\omega)$ for some $\eta > 1$, then the asymptotic convergence factor of (1.9) is less or equal to $1/\eta$, whereas that of the SOR method is less or equal to $1/\eta^p$. In Section 3, we show that the boundaries of the regions $U(\omega)$ and $U_\eta(\omega)$ are given by different types of hypocycloids.

For ease of notation, let us introduce—for a given $p \in \mathbb{N}$ with $p \geqslant 2$—the "stars"

$$S_\beta^+ := \left\{ z \in \mathbb{C} : z = \mu e^{2k\pi i/p},\ 0 \leqslant \mu \leqslant \beta,\ k = 0,1,\dots,p-1 \right\} \tag{1.11a}$$

and

$$S_\beta^- := \left\{ z \in \mathbb{C} : z = \mu e^{(2k+1)\pi i/p},\ 0 \leqslant \mu \leqslant \beta,\ k = 0,1,\dots,p-1 \right\}. \tag{1.11b}$$

Clearly, S_β^- results from S_β^+ by a rotation of π/p; Figure 1 shows S_β^+ and S_β^- for $p = 5$.

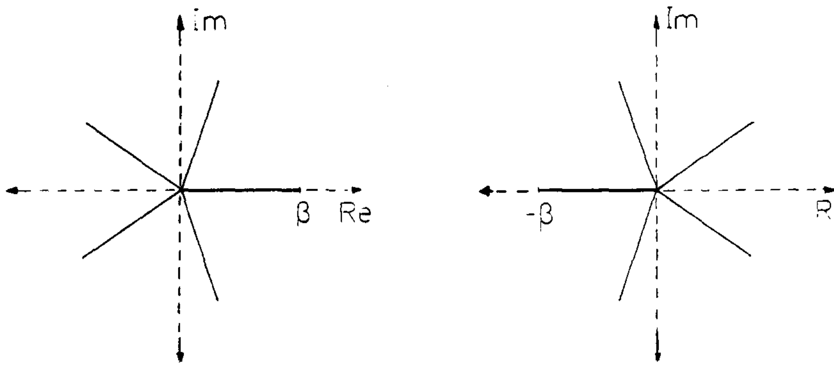

FIG. 1.

Now the following lemma follows directly from the properties of a weakly cyclic matrix of index p (see Varga [17, p. 97 ff.]).

LEMMA 1. *Let the Jacobi matrix J_A be weakly cyclic of index p with $\beta := \varrho(J_A)$. Then the eigenvalues of J_A^p are nonnegative (nonpositive) iff the eigenvalues of J_A are contained in S_β^+ (S_β^-).*

Thus, the determination of the exact convergence intervals in Section 4 means: Determine—for a fixed J_A with $\sigma(J_A) \subset S_\beta^+$ (respectively S_β^-)—all values of $\omega \in \mathbb{R}$ such that S_β^+ (respectively S_β^-) is contained in $U(\omega)$; these intervals are given in terms of Chebyshev polynomials. Finding the optimal relaxation factor ω_{opt} in Section 5 means: Find ω_{opt} such that S_β^+ (respectively S_β^-) is contained in $U_\eta(\omega_{\text{opt}})$ and $1/\eta$ becomes minimal for $\omega = \omega_{\text{opt}}$. The fact that overrelaxation is more appropriate if the eigenvalues of J_A^p are nonnegative is reflected by the fact that the regions $U_\eta(\omega)$ fit better to S_β^+ than to S_β^- for $\omega > 1$, and conversely for $\omega < 1$. (See also remark (1) in Section 6 below.)

2. THE p-STEP RELAXATION METHOD

For a given linear system

$$\mathbf{x} = J_A \mathbf{x} + \mathbf{c} \tag{2.1}$$

with weakly cyclic matrix J_A of index p [in normal form (1.4)] we consider the p-step relaxation

$$\mathbf{x}^{(m)} = \omega J_A \mathbf{x}^{(m-1)} + (1-\omega)\mathbf{x}^{(m-p)} + \omega \mathbf{c}, \qquad m = p, p+1, \ldots. \tag{2.2}$$

If the vectors $\mathbf{x}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_p^{(m)})^T$ and $\mathbf{c} = (c_1, c_2, \ldots, c_p)^T$ are partitioned according to the partitioning of the matrix J_A in (1.4), we find

$$\begin{aligned} x_1^{(m)} &= \omega B_1 x_p^{(m-1)} + (1-\omega) x_1^{(m-p)} + \omega c_1, \\ x_2^{(m+1)} &= \omega B_2 x_1^{(m)} + (1-\omega) x_2^{(m-p+1)} + \omega c_2, \\ &\;\vdots \\ x_p^{(m+p-1)} &= \omega B_p x_{p-1}^{(m+p-2)} + (1-\omega) x_p^{(m-1)} + \omega c_p. \end{aligned} \tag{2.3}$$

On the other hand, if we apply one step of SOR (1.2) to the auxiliary vector

$$\mathbf{y}^{(m-1)} := \left(x_1^{(m-p)}, x_2^{(m-p+1)}, \dots, x_p^{(m-1)}\right)^T, \tag{2.4}$$

we get

$$\mathbf{y}^{(m+p-1)} := \left(x_1^{(m)}, x_2^{(m+1)}, \dots, x_p^{(m+p-1)}\right)^T, \tag{2.5}$$

i.e., the block components of $\mathbf{y}^{(m-1)}$, after *one* SOR step, are precisely the same as those obtained after p steps of the iteration (2.2). Thus, we have

THEOREM 1. *The p-step relaxation method* (2.2), *if applied to a linear system* (2.1) *with weakly cyclic Jacobi matrix* J_A *of index* p, *converges, for arbitrary starting vectors, to the solution* $\mathbf{x}$ *of* (2.1) *if and only if the* SOR *method* (1.2) *converges. In case of convergence, the* SOR *method converges* p *times faster than* (2.2).

By this observation, we can restrict ourselves to the easier study of the iteration (2.2) instead of the SOR method (1.2). We remark that (2.2) is a special kind of a semiiterative method, i.e., $\mathbf{x}^{(m)}$ can be interpreted as a "weighted average" of the vector $\tilde{\mathbf{x}}^{(m)} = J_A\mathbf{x}^{(m-1)} + \mathbf{c}$, obtained by the usual Jacobi iteration, and the vector $\mathbf{x}^{(m-p)}$. Such methods have been extensively studied in Niethammer and Varga [10], Eiermann and Niethammer [2], and Eiermann, Niethammer, and Varga [3]. By applying results derived from summability theory, it therefore would be possible to obtain statements about convergence and optimality of (2.2). However, we shall use a more straightforward method to examine the p-step relaxation (2.2).

Following Gutknecht, Niethammer, and Varga [5], the pth-order difference equation (2.2) can be transformed into an equivalent first-order difference equation

$$\mathbf{x}^{(m)} := \mathbf{T}\mathbf{x}^{(m-1)} + \omega\mathbf{c}, \quad m = p, p+1, \dots,$$

$$\text{where} \quad \mathbf{x}^{(m)} \in \mathbb{R}^{np} \text{ and } \mathbf{T} \in \mathbb{R}^{np,np}. \tag{2.6}$$

Again, as in (1.2), the iteration (2.6) converges for arbitrary starting vectors iff for the spectral radius $\varrho(\mathbf{T})$ of $\mathbf{T}$, there holds $\varrho(\mathbf{T}) < 1$; the asymptotic convergence factor of (2.6) is given by $\varrho(\mathbf{T})$. Now the spectrum of $\mathbf{T}$ is related to the spectrum of J_A by the following well-known result (see, e.g., Rjabenki and Filippow [14]): If μ_i $(i = 1, \dots, n)$ are the eigenvalues of J_A, then the eigenvalues of $\mathbf{T}$ are the np zeros of the n polynomials

$$p_j(\lambda) := \lambda^p - \omega\mu_j\lambda^{p-1} - (1-\omega) \qquad (j = 1,\dots,n). \tag{2.7}$$

Now, the zeros of all polynomials p_j are in the open unit disk, i.e., $\varrho(\mathbf{T}) < 1$, iff $p_j(\lambda) \neq 0$ for $|\lambda| \geqslant 1$ and $j = 1,\dots,n$. By considering the reciprocal polynomial, i.e., $\lambda^p p_j(1/\lambda)$ instead of $p_j(\lambda)$, this is equivalent to

$$\mu_j \neq \frac{1-(1-\omega)\lambda^p}{\omega\lambda} =: q_\omega(\lambda) \qquad \text{for} \quad |\lambda| \leqslant 1 \text{ and } j = 1,\dots,n. \tag{2.8}$$

In the same way, we conclude $\varrho(\mathbf{T}) \leqslant 1/\eta$ iff $\mu_j \neq q_\omega(\lambda)$ for $|\lambda| < \eta$ $(j = 1,\dots,n)$ and some $\eta > 1$. Thus, we have sketched the main idea of the proof of the following theorem (see [5, Theorem 2] for the general case).

THEOREM 2. *The p-step relaxation* (2.2) *and the SOR method* (1.2) *applied to a linear system with a weakly cyclic Jacobi matrix* J_A *of index* p *converge for arbitrary starting vectors iff the spectrum* $\sigma(J_A)$ *is contained in*

$$U(\omega) := \mathbb{C} \setminus q_\omega(\overline{D}_1), \tag{2.9}$$

where q_ω *is defined in* (2.8) *and* $\overline{D}_1 := \{z \in \mathbb{C} : |z| \leqslant 1\}$. *The asymptotic convergence factor of* (2.2) *is at most* $1/\eta < 1$ *if* $\sigma(J_A)$ *is contained in*

$$U_\eta(\omega) := \mathbb{C} \setminus q_\omega(D_\eta), \tag{2.10}$$

where $D_\eta := \{z \in \mathbb{C} : |z| < \eta\}$. *The asymptotic convergence factor is exactly* $1/\eta$ *if at least one eigenvalue of* J_A *is on the boundary of* $U_\eta(\omega)$.

REMARKS.

(1) Depending on ω and η, the set $U_\eta(\omega)$ may be empty.

(2) In view of Theorem 1, for the SOR operator L_ω there holds $\varrho(L_\omega) \leqslant 1/\eta^p$ if $\sigma(J_A) \subset U_\eta(\omega)$.

(3) By using Theorem 1, Varga's formula (1.5) can be derived easily from (2.7).

3. GEOMETRIC DESCRIPTION OF THE SETS $U(\omega)$ AND $U_\eta(\omega)$

To derive convergence results for the SOR method from Theorem 2, we have to describe the sets $U(\omega)$ and $U_\eta(\omega)$ first. We do this by studying the image curves of the circles $\partial D_\eta := \{ z \in \mathbb{C} : |z| = \eta \}$ under the mapping q_ω of (2.8); the boundaries of $U(\omega)$ and $U_\eta(\omega)$ consist of parts of these curves. Depending on ω and η, there will appear three types:

(1) shortened hypocycloids,
(2) ordinary or cusped hypocycloids,
(3) stretched hypocycloids.

These curves are the locus of a point P inside [case (1)], on [case (2)], or outside [case (3)] a circle with midpoint M and radius r which rolls without sliding on the inside of a fixed circle with radius R (cf. Salmon [15], Fladt [4, p. 322]). The distance between P and M is given by h. Clearly we have $h < r$, $h = r$, $h > r$ in cases (1), (2), (3) respectively. By letting $\lambda = \eta e^{i\theta}$, $0 \leqslant \theta < 2\pi$, in (2.8), we find

$$z := q_\omega(\lambda) = x + iy, \tag{3.1}$$

where

$$x = \frac{1}{\omega}\left(\frac{1}{\eta}\cos\theta + (\omega - 1)\eta^{p-1}\cos(p-1)\theta\right), \tag{3.2a}$$

$$y = -\frac{1}{\omega}\left(\frac{1}{\eta}\sin\theta - (\omega - 1)\eta^{p-1}\sin(p-1)\theta\right). \tag{3.2b}$$

Let $\omega > 1$. Comparing (3.2) with the parametric representation in Fladt [4, p. 324], it is now an easy task to show that $q_\omega(\partial D_\eta)$ is within the class of hypocycloidal curves. We just have to put

$$r = \frac{1}{(p-1)\omega\eta}, \qquad R = \frac{p}{(p-1)\omega\eta}, \qquad h = \frac{\omega - 1}{\omega}\eta^{p-1}.$$

We note that the ratio R/r always equals p, i.e., the curve consists of p congruent arcs.

For our later investigations on the convergence of the iteration (2.2), we have to know the intersection points of $q_\omega(\partial D_\eta)$ with the positive real axis and the axis given by $\arg z = \pi/p$. The latter ones are obtained by setting

TABLE 1

ω	η[a]	Geometrical classification	Intersection points with pos. real axis	Intersection points with axis $\arg z = \pi/p$
$1<\omega<\frac{p}{p-1}$	$1\leqslant\eta<\tilde{\eta}$	Case 1	1 point: abscissa $\frac{1}{\omega}\left(\frac{1}{\eta}+(\omega-1)\eta^{p-1}\right)$	1 point: modulus $\frac{1}{\omega}\left(\frac{1}{\eta}-(\omega-1)\eta^{p-1}\right)$
	$\eta=\tilde{\eta}$	Case 2 (interval on real axis if $p=2$)	1 point (endpoint of the interval if $p=2$): abscissa $\frac{p}{p-1}\frac{1}{\omega}\sqrt[p]{(p-1)(\omega-1)}$	As above (if $p\geqslant 3$)
	$\tilde{\eta}<\eta<\hat{\eta}$ ($p\geqslant 3$)	Case 3	2 points	As above
$\omega=\frac{p}{p-1}$ (if $p\geqslant 3$)	$\eta=\tilde{\eta}=1$	Case 2	1 point: abscissa 1	As above
	$1<\eta<\hat{\eta}$	Case 3	2 points	As above
$\frac{p}{p-1}<\omega<2$ (if $p\geqslant 3$)	$1\leqslant\eta<\hat{\eta}$	Case 3	2 points	As above

[a]Notation: $\tilde{\eta}=\{(p-1)(\omega-1)\}^{-1/p}$, $\hat{\eta}=(\omega-1)^{-1/p}$.

$\theta=-\pi/p$ in (3.2a), (3.2b). We summarize the essential geometric results in Table 1 [see Remark (3) near the end of this section].

Now, let $\omega<1$. We rotate the curve given by (3.2a), (3.2b) by the angle π/p, i.e., instead of (3.1) we consider

$$z'=ze^{i\pi/p}=\frac{1-(1-\omega)\eta^pe^{ip\theta}}{\omega\eta e^{i\theta}}e^{i\pi/p}=\frac{1+(1-\omega)\eta^pe^{ip(\theta-\pi/p)}}{\omega\eta e^{i(\theta-\pi/p)}}$$

$$=\frac{1+(1-\omega)\eta^pe^{ip\theta'}}{\omega\eta e^{i\theta'}}, \tag{3.3}$$

where $\theta'=\theta-\pi/p$. Now, z' is again within the class of hypocycloidal

TABLE 2

ω	η[a]	Geometrical classification	Intersection points with pos. real axis	Intersection points with axis $\arg z = \pi/p$
$0 < \omega < \frac{p-2}{p-1}$ (if $p \geqslant 3$)	$1 \leqslant \eta < \hat{\eta}$	Case 3 (rot. by π/p)	1 point: abscissa $\frac{1}{\omega}\left(\frac{1}{\eta} - (1-\omega)\eta^{p-1}\right)$	2 points
$\omega = \frac{p-2}{p-1}$ (if $p \geqslant 3$)	$\eta = \tilde{\eta} = 1$	Case 2 (rot. by π/p)	1 point: abscissa 1	1 point: modulus $p/(p-2)$
	$1 < \eta < \hat{\eta}$	Case 3 (rot. by π/p)	1 point: $\frac{1}{\omega}\left(\frac{1}{\eta} - (1-\omega)\eta^{p-1}\right)$	2 points
$\frac{p-2}{p-1} < \omega < 1$	$1 \leqslant \eta < \tilde{\eta}$	Case 1 (rot. by π/p)	As above	1 point: modulus $\frac{1}{\omega}\left(\frac{1}{\eta} + (1-\omega)\eta^{p-1}\right)$
	$\eta = \tilde{\eta}$	Case 2 (rot. by π/p)[b]	As above (if $p \geqslant 3$)	1 point (endpoint of interval if $p = 2$): modulus $\frac{p}{p-1}\frac{1}{\omega}\sqrt[p]{(p-1)(1-\omega)}$
	$\tilde{\eta} < \eta < \hat{\eta}$ ($p \geqslant 3$)	Case 3 (rot. by π/p)	As above	2 points

[a] Notation: $\tilde{\eta} = \{(p-1)(1-\omega)\}^{-1/p}$, $\hat{\eta} = (1-\omega)^{-1/p}$.
[b] Interval on the imaginary axis if $p = 2$.

curves, i.e., z is a rotated hypocycloidal curve. As in the case $\omega > 1$, it is easy to obtain the results shown in Table 2.

REMARKS.

(1) The case $\omega = 1$ could be neglected in Tables 1, 2 because for $\omega = 1$ the iteration (1.2) as well as (2.2) degenerates to the simple Jacobi iteration, which converges iff $\varrho(J_A) < 1$.

(2) Note that for $\eta \geqslant \hat{\eta}$ (if $p \geqslant 3$), the sets $U_\eta(\omega)$ are empty [see Remark (1) following Theorem 2].

(3) It should be mentioned that the case $p = 2$ plays an exceptional role in the considerations above. Here for $1 < \omega < 2$ $(0 < \omega < 1)$ and $1 \leqslant \eta < \tilde{\eta} = 1/\sqrt{|\omega - 1|}$, the curves $q_\omega(\partial D_\eta)$ are ellipses with foci on the real (imaginary) axis, whereas for $\eta = \tilde{\eta}$, the curves collapse to an interval on the real (imaginary) axis, and for $\eta > \tilde{\eta}$, the sets $U_\eta(\omega)$ are empty. The ellipses represent shortened hypocycloids, while intervals represent cusped hypocycloids in the case $p = 2$.

In Figure 2, we have plotted the three types of curves for both cases $\omega < 1$ and $\omega > 1$, where $p = 5$.

With this simple geometrical classification (Tables 1,2) in mind, it is now easy to investigate the convergence of the p-step relaxation method (2.2).

4. THE EXACT INTERVALS OF CONVERGENCE

4.1. *Eigenvalues of J_A^p Nonnegative*

We assume that the spectrum $\sigma(J_A^p)$ of J_A^p is contained in the interval $[0, \beta^p]$, where $\beta = \varrho(J_A)$ denotes the spectral radius of J_A. By Lemma 1, it follows that $\sigma(J_A)$ is contained in the star S_β^+ [see (1.11a)]. By Theorem 2, we have to examine whether S_β^+ is contained in $U(\omega)$. For this, it is sufficient to determine the intersection points of the boundary $\partial U(\omega)$ with the positive real axis. From Tables 1,2, we have the following cases:

(i) $0 < \omega < p/(p-1)$: There is only one intersection point of $q_\omega(\partial D_1)$. The associated value of the abscissa is 1, so that we have convergence of (2.2) for any $\beta = \varrho(J_A)$ satisfying $\beta < 1$. If $p = 2$, we have the well-known result that the SOR method converges for any

$$0 < \omega < 2, \qquad 0 \leqslant \beta < 1$$

(cf. Varga [16]).

(ii) $\omega = p/(p-1)$, $p \geqslant 3$: $q_\omega(\partial D_1)$ is a cusped hypocycloid (case 2), and we get a cusp on the positive real axis with abscissa 1. As in (i), we obtain convergence of (2.2) for any $\beta = \varrho(J_A) < 1$.

(iii) $p/(p-1) < \omega < 2$, $p \geqslant 3$: $q_\omega(\partial D_1)$ is a stretched hypocycloid (case 3), and there is, besides the point 1, a second intersection point of the curve with the positive real axis. The loops (see Figure 2 below) at the axes $\arg z = 2\pi k/p$, $k = 0, 1, \ldots, p-1$, do not belong to $U(\omega)$, i.e., the points

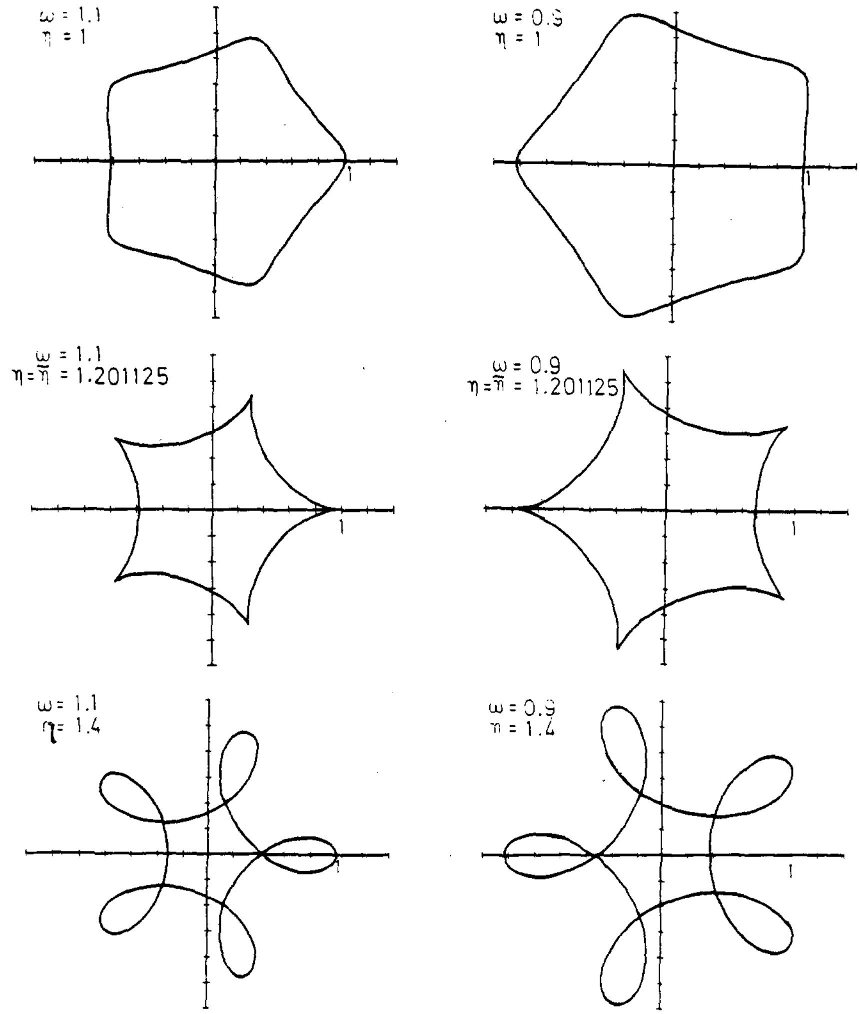

FIG. 2. Curves $q_\omega(\partial D_\eta)$, $p = 5$.

within and on the loops occur as images of points in $\overline{D}_1$ under q_ω. Therefore, we need the abscissa of the smaller point of intersection with the positive real axis.

From (3.2b), we find

$$y = \frac{1}{\omega}\left[\sin\theta + (\omega - 1)\sin(p-1)\theta\right] \stackrel{!}{=} 0. \tag{4.1}$$

$\theta = 0$ yields the outer point of intersection with abscissa 1. For the inner one, we must have

$$\frac{\sin(p-1)\theta}{\sin\theta} = \frac{1}{\omega - 1} \qquad \text{with a } \theta \in (0, \pi/p). \tag{4.2}$$

If U_{p-2} denotes the $(p-2)$th-degree Chebyshev polynomial of the second kind (cf. Rivlin [13]), then by substituting $t = \cos\theta$ in (4.2), we get

$$U_{p-2}(t) - \frac{1}{\omega - 1} = 0, \tag{4.3}$$

which has to be solved for a $t \in (\cos(\pi/p), 1)$. The corresponding value of the abscissa is then given by

$$x = \frac{1}{\omega}\left[\cos\theta + (\omega - 1)\cos(p-1)\theta\right] = \frac{1}{\omega}\left[t + (\omega - 1)T_{p-1}(t)\right], \tag{4.4}$$

where T_{p-1} denotes the $(p-1)$th-degree Chebyshev polynomial of the first kind (cf. Rivlin [13]). To obtain convergence of (2.2) in this case, we must have $\beta < x$, x given by (4.4). Thus, we have:

THEOREM 3. *Let the block Jacobi matrix J_A associated with the linear system $A\mathbf{x} = \mathbf{b}$ be weakly cyclic of index p, let the eigenvalues of J_A^p be nonnegative, and let $\beta = \varrho(J_A)$ denote the spectral radius of J_A. Then:*

(1) *The* SOR *method* (1.2) *converges for any $\beta < 1$ if $0 < \omega < 2$, $p = 2$ or if $0 < \omega \leqslant p/(p-1)$, $p \geqslant 3$.*

(2) *If $p \geqslant 3$ and if $p/(p-1) < \omega < 2$, the only values of ω which give convergence are such that β satisfies*

$$\beta < \frac{1}{\omega}\left[t + (\omega - 1)T_{p-1}(t)\right],$$

where $t \in (\cos(\pi/p), 1)$ is the unique solution of

$$U_{p-2}(t) = \frac{1}{\omega - 1}.$$

Here T_{p-1}, U_{p-2} denote Chebyshev polynomials of the first and the second kind of degree $p-1$ and $p-2$, respectively.

We treat some special cases. For notation, let $\hat{\Omega}_p$ be the set of pairs (ω, β) for which the p-step relaxation (2.2) and equivalently the SOR method are convergent (cf. Hadjidimos, Li, and Varga [6]).

By some elementary computation, we find

$$\hat{\Omega}_2 = \{(\omega, \beta): 0 < \omega < 2, 0 \leqslant \beta < 1\}$$

(see Varga [16]),

$$\hat{\Omega}_3 = \{(\omega, \beta): 0 < \omega < \hat{\omega}_3(\beta), 0 \leqslant \beta < 1\}, \qquad \text{where} \quad \hat{\omega}_3(\beta) = \frac{\beta + 2}{\beta + 1}$$

(see Niethammer, de Pillis, and Varga [11]),

$$\hat{\Omega}_4 = \{(\omega, \beta): 0 < \omega < \hat{\omega}_4(\beta), 0 \leqslant \beta < 1\},$$

where

$$\hat{\omega}_4(\beta) = \frac{4 - \beta^2 - \beta\sqrt{\beta^2 + 8}}{2(1 - \beta^2)},$$

and

$$\hat{\Omega}_5 = \{(\omega, \beta): 0 < \omega < \hat{\omega}_5(\beta), 0 \leqslant \beta < 1\},$$

where

$$\hat{\omega}_5(\beta) = \frac{4 - \beta - \beta^2 - \beta\sqrt{5 - 2\beta + \beta^2}}{2(1 - \beta^2)}.$$

Figure 3 shows the sets $\hat{\Omega}_2$, $\hat{\Omega}_3$, $\hat{\Omega}_4$, and $\hat{\Omega}_5$.

4.2. *Eigenvalues of J_A^p Nonpositive*

Let us now assume that all eigenvalues of J_A^p are nonpositive, i.e., the eigenvalues of J_A are contained in S_β^-, where $\beta := \varrho(J_A)$ [see (1.11b)]. To investigate convergence, we are now interested in points of intersection of $q_\omega(\partial D_1)$ with the line $\arg z = \pi/p$.

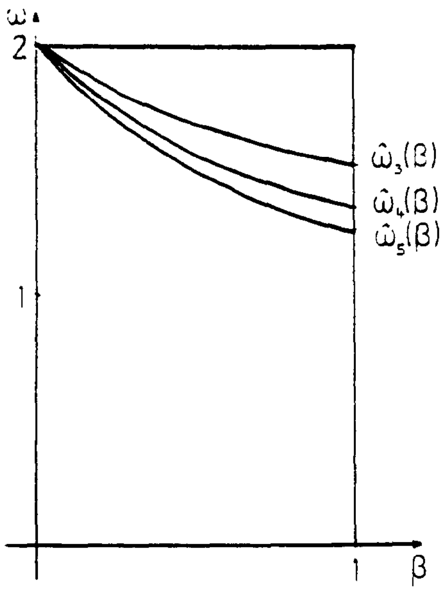

FIG. 3.

From Tables 1.2, we have the following cases:

(i) $(p-2)/(p-1) < \omega < 2$: There is only one point of intersection of $q_\omega(\partial D_1)$ with $\arg z = \pi/p$. Its absolute value is always given by $(2-\omega)/\omega$. Thus, the p-step relaxation converges if $\beta < (2-\omega)/\omega$, i.e. $\omega < 2/(\beta+1)$. The value on the right-hand side of the last inequality is always positive and less or equal to 2, so that we have no further restriction on ω and β if $p = 2$. We have the result that in this case the SOR method converges for $0 < \omega < 2/(\beta+1)$, $0 \leqslant \beta < \infty$ (cf. Niethammer [9]). The discussion for $p = 2$ is finished. Let us assume $p \geqslant 3$. Since ω is restricted to be greater than $(p-2)/(p-1)$, we have

$$\frac{p-2}{p-1} \leqslant \omega < \frac{2}{\beta+1} \quad \text{and} \quad \beta < \frac{p}{p-2}. \tag{4.5}$$

(ii) $\omega = (p-2)/(p-1)$, $p \geqslant 3$: Here $q_\omega(\partial D_1)$ is a (rotated) cusped hypocycloid and we get a cusp on the axis $\arg z = \pi/p$ with modulus $(2-\omega)/\omega$. As in (i), we obtain convergence of (2.2) for $\beta < (2-\omega)/\omega = p/(p-2)$.

(iii) $0 < \omega < (p-2)/(p-1)$, $p \geqslant 3$: In this case, $q_\omega(\partial D_1)$ is a (rotated) stretched hypocycloid having two points of intersection with $\arg z = \pi/p$. After a rotation of $q_\omega(\partial D_1)$ by the angle π/p as in Section 4.1, we find the condition

$$\frac{\sin(p-1)\theta'}{\sin\theta'} = \frac{1}{1-\omega} \qquad \text{for some} \quad \theta' \in (0, \pi/p).$$

Again substituting $t' = \cos\theta'$, this is equivalent to

$$U_{p-2}(t') - \frac{1}{1-\omega} = 0, \qquad t' \in \left(\cos\frac{\pi}{p}, 1\right). \tag{4.6}$$

The corresponding value of the abscissa is then given by

$$x = \frac{1}{\omega}\left[t' + (1-\omega)T_{p-1}(t')\right], \tag{4.7}$$

and SOR is only convergent if $\beta = \varrho(J_A)$ satisfies $\beta < x'$.

THEOREM 4. *Let the Jacobi matrix J_A associated with the linear system $A\mathbf{x} = \mathbf{b}$ be weakly cyclic of index p, let the eigenvalues of J_A^p be nonpositive, and let $\beta = \varrho(J_A)$ denote the spectral radius of J_A. Then:*

(1) *The* SOR *method converges for any $0 \leqslant \beta < \infty$ if $0 < \omega < 2/(\beta+1)$, $p = 2$, or for any $\beta < p/(p-2)$ if $(p-2)/(p-1) \leqslant \omega < 2/(\beta+1)$, $p \geqslant 3$.*

(2) *If $p \geqslant 3$ and if $0 < \omega < (p-2)/(p-1)$, the only values of ω which give convergence are such that β satisfies*

$$\beta < \frac{1}{\omega}\left[t' + (1-\omega)T_{p-1}(t')\right],$$

where $t' \in (\cos(\pi/p), 1)$ is the solution of $U_{p-2}(t') = 1/(1-\omega)$.

As in Theorem 3, T_{p-1} and U_{p-2} denote Chebyshev polynomials of the first and second kind, respectively.

Let us again consider some special cases. Now Ω_p denotes the set of pairs (ω, β) for which the p-step relaxation and equivalently the SOR method are convergent (cf. [6]). As in Section 4.1, we find

$$\Omega_2 = \left\{(\omega, \beta) : 0 < \omega < \frac{2}{\beta+1} \text{ if } 0 \leqslant \beta < \infty\right\}$$

(see Niethammer [9]);

$$\Omega_3 = \left\{ (\omega,\beta) : 0 < \omega < \frac{2}{\beta+1} \text{ if } 0 \leqslant \beta < 2, \right.$$

$$\left. \text{and } \omega_3(\beta) < \omega < \frac{2}{\beta+1} \text{ if } 2 \leqslant \beta < 3 \right\},$$

where

$$\omega_3(\beta) = \frac{\beta-2}{\beta-1}$$

(see Niethammer, de Pillis, and Varga [11]);

$$\Omega_4 = \left\{ (\omega,\beta) : 0 < \omega < \frac{2}{\beta+1} \text{ if } 0 \leqslant \beta < \sqrt{2}, \right.$$

$$\left. \text{and } \omega_4(\beta) < \omega < \frac{2}{\beta+1} \text{ if } \sqrt{2} \leqslant \beta < 2 \right\},$$

where

$$\omega_4(\beta) = \frac{\beta^2-2}{\beta^2-1};$$

$$\Omega_5 = \left\{ (\omega,\beta) : 0 < \omega < \frac{2}{\beta+1} \text{ if } 0 \leqslant \beta < \sqrt{5}-1, \right.$$

$$\left. \text{and } \omega_5(\beta) < \omega < \frac{2}{1+\beta} \text{ if } \sqrt{5}-1 \leqslant \beta < \tfrac{5}{3} \right\},$$

where

$$\omega_5(\beta) = \frac{\beta^2-\beta-4+\beta\sqrt{\beta^2+2\beta+5}}{2(\beta^2-1)};$$

$$\Omega_6 = \left\{ (\omega,\beta) : 0 < \omega < \frac{2}{\beta+1} \text{ if } 0 \leqslant \beta < \sqrt{\tfrac{4}{3}}, \right.$$

$$\left. \text{and } \omega_6(\beta) < \omega < \frac{2}{\beta+1} \text{ if } \sqrt{\tfrac{4}{3}} \leqslant \beta < \tfrac{3}{2} \right\},$$

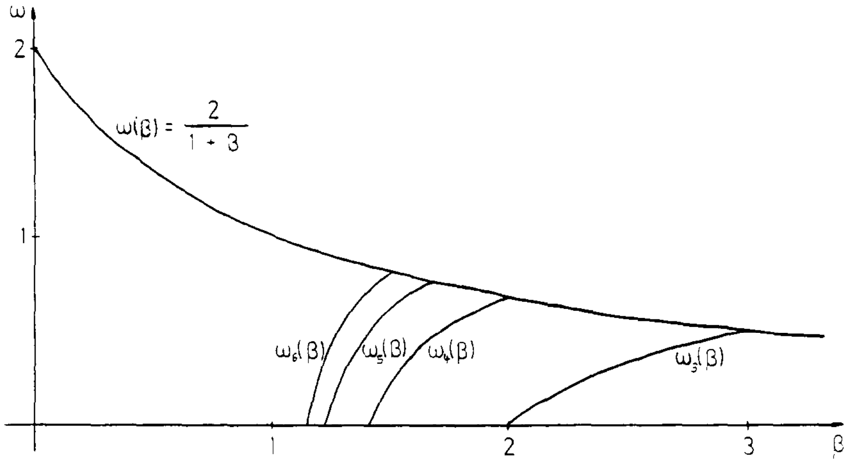

FIG. 4.

where

$$\omega_6(\beta)=\frac{\beta^2+\beta\sqrt{\beta^2+4}-4}{2(\beta^2-1)}.$$

Figure 4 shows the sets Ω_2, Ω_3, Ω_4, Ω_5, and Ω_6.

5. THE OPTIMAL RELAXATION PARAMETER

For $\beta=\varrho(J_A)=0$, the stars S_β^+ and S_β^- [see (1.11a,b)] degenerate to the point $z=0$. Since some of our conclusions cannot be drawn in this case, let us assume henceforth $\varrho(J_A)>0$. For $\varrho(J_A)=0$, there holds $\omega_{\text{opt}}=1$ by a continuity argument.

5.1. *Eigenvalues of J_A^p Nonnegative*

Let a weakly cyclic Jacobi matrix of index p with $0<\beta=\varrho(J_A)<1$ be given. As can be seen from Table 1 for $1<\omega<p/(p-1)$, there is exactly one value of η, namely $\eta=\tilde{\eta}=1/\sqrt[p]{(p-1)(\omega-1)}$, such that $q_\omega(\partial D_\eta)$ is a cusped hypocycloid (an interval on the real axis if $p=2$).

We first show that there is a unique $\omega^*\in(1,p/(p-1))$ such that the corresponding cusped hypocycloid $q_{\omega^*}(D_\eta)$ where $\eta=\tilde{\eta}^*=$

$1/\sqrt[p]{(p-1)(\omega^*-1)}$ will pass through the point β (or that β is one endpoint of the interval if $p=2$). The point of intersection of the cusped hypocycloid with the nonnegative real axis (positive endpoint of the interval if $p=2$) is given by

$$\frac{p}{(p-1)\omega}\sqrt[p]{(p-1)(\omega-1)}\,.$$

This value should equal β; thus, we get

$$\frac{p^p}{(p-1)^{p-1}}(\omega-1)=\beta^p\omega^p, \tag{5.1}$$

or equivalently (cf. Varga [17, p. 109])

$$(p-1)^{p-1}\omega^p\beta^p-(\omega-1)p^p=0; \tag{5.2}$$

(5.2) has to be satisfied by ω^*. By easy analytic arguments, we find the following

Result. *There is a unique* $\omega^*\in(1,p/(p-1))$ *determined by* (5.2). *The associated p-step relaxation method converges with the asymptotic rate*

$$\kappa=\frac{1}{\tilde{\eta}^*}=\sqrt[p]{(p-1)(\omega^*-1)}\,. \tag{5.3}$$

We now assert that this is the best possible asymptotic convergence factor, i.e., ω^* is the optimal relaxation parameter. For simplicity, we assume $p\geqslant 3$. However, the case $p=2$ could be treated by slight modifications, and the main results (Theorem 5 and 6) also hold if $p=2$. In order to prove the assertion for $p\geqslant 3$, we shall distinguish two cases for ω and show that either $\tilde{\eta}^*\geqslant\tilde{\eta}(\omega)$ or $q_\omega(\partial D_{\tilde{\eta}^*})$ has a smaller (inner) point of intersection with the positive real axis than β. In both cases the corresponding p-step relaxation (when performed with ω) can never yield a convergence factor $\leqslant 1/\tilde{\eta}^*$.

(i) $0<\omega<\omega^*$: Since we must have $\tilde{\eta}^*<\hat{\eta}=1/\sqrt[p]{|1-\omega|}$, we only need to consider those ω satisfying $\omega>1-(p-1)(\omega^*-1)$. But from Tables 1.2, for these ω there is only one point of intersection of $q_\omega(\partial D_{\tilde{\eta}^*})$ with the positive real axis. Its abscissa is given by

$$\frac{1}{\omega}\left(\frac{1}{\tilde{\eta}^*}+(\omega-1)\tilde{\eta}^{*p-1}\right)=: g(\omega).$$

By differentiation, we find that $g(\omega)$ is strictly monotonic increasing, i.e., $g(\omega^*)=\beta > g(\omega)$ for $\omega<\omega^*$, and the p-step relaxation, when performed with ω, cannot yield the convergence factor $1/\tilde{\eta}^*$.

(ii) $\omega^*<\omega<2$: Again we only need to consider those values of ω where $\tilde{\eta}^*<\eta<\hat{\eta}=1/\sqrt[p]{\omega-1}$, i.e., we must have $\omega<1+(p-1)(\omega^*-1)$. Because

$$\tilde{\eta}^*=\frac{1}{\sqrt[p]{(p-1)(\omega^*-1)}}>\frac{1}{\sqrt[p]{(p-1)(\omega-1)}}=\tilde{\eta},$$

the curve $q_\omega(\partial D_{\tilde{\eta}^*})$ has two points of intersection with the positive real axis. The smaller one is given by

$$x=\frac{1}{\omega}\left(\frac{1}{\tilde{\eta}^*}t+(\omega-1)\tilde{\eta}^{*p-1}T_{p-1}(t)\right), \tag{5.4}$$

where $t\in(\cos(\pi/p),1)$ is the solution of $U_{p-2}(t)=1/[(\omega-1)\tilde{\eta}^{*p}]$. We have to prove

$$x<x^*=\frac{1}{\omega}\left(\frac{1}{\tilde{\eta}^*}t^*+(\omega-1)\tilde{\eta}^{*p-1}T_{p-1}(t^*)\right),\qquad \text{where}\quad t^*=1. \tag{5.5}$$

[$x^*=\beta$ gives the value of the abscissa of the cusp of $q_{\omega^*}(\partial D_{\tilde{\eta}^*})$.] Since $t<1$, we have $(1/\tilde{\eta}^*)t<1/\tilde{\eta}^*$. Thus, it is sufficient to prove (note $\omega>\omega^*$)

$$(\omega-1)\tilde{\eta}^{*p-1}T_{p-1}(t)\leqslant(\omega^*-1)\tilde{\eta}^{*p-1}, \tag{5.6}$$

or equivalently,

$$T_{p-1}(t)\leqslant\frac{\omega^*-1}{\omega-1}, \tag{5.7}$$

where t satisfies $U_{p-2}(t)=1/[(\omega-1)\tilde{\eta}^{*p}]=(p-1)(\omega^*-1)/(\omega-1)$. Thus, in order to prove (5.7), we show

$$U_{p-2}(t) \geqslant (p-1)T_{p-1}(t) \qquad \text{for} \quad t \in (\cos(\pi/p), 1). \tag{5.8}$$

But this follows easily from properties of the Chebyshev polynomials and is not argued here.

THEOREM 5. *Let the Jacobi matrix associated with the linear system* $A\mathbf{x} = \mathbf{b}$ *be weakly cyclic of index* p, *and let the eigenvalues of* J_A^p *be nonnegative with*

$$0 < \beta = \varrho(J_A) < 1.$$

The optimal relaxation parameter ω_{opt} *of the* p*-step relaxation and equivalently of the* SOR *method is given by the (unique) solution* $\omega^* \in (1, p/(p-1))$ *of the equation*

$$(p-1)^{p-1}\omega^p\beta^p - p^p(1-\omega) = 0.$$

The corresponding p*-step relaxation method converges with an asymptotic convergence factor*

$$\frac{1}{\tilde{\eta}^*} = \sqrt[p]{(p-1)(\omega^*-1)},$$

and thus for the SOR *method we have*

$$\varrho(L_\omega^*) = (p-1)(\omega^*-1).$$

5.2. *Eigenvalues of* J_A^p *Nonpositive*

By a proof analogous to the one given above, we find

THEOREM 6. *Let the Jacobi matrix associated with the linear system* $A\mathbf{x} = \mathbf{b}$ *be weakly cyclic of index* p, *and let the eigenvalues of* J_A^p *be nonpositive with*

$$0 < \beta = \varrho(J_A) < \frac{p}{p-2} \qquad (0 < \beta < \infty \quad \text{if } p = 2).$$

The optimal relaxation parameter ω_{opt} *of the p-step relaxation and equivalently of the* SOR *method, is given by the (unique) solution* $\omega^* \in (p - 1/p - 2, 1)$ *of the equation*

$$(p-1)^{p-1}\omega^p\beta^p - p^p(1-\omega) = 0.$$

The corresponding p-step relaxation method converges with an asymptotic convergence factor

$$\frac{1}{\tilde{\eta}^*} = \sqrt[p]{(p-1)(1-\omega^*)},$$

and thus for the SOR *method we have*

$$\varrho(L_{\omega^*}) = (p-1)(1-\omega^*).$$

6. FINAL REMARKS

(1) By an easy calculation [compute $|q_\omega(\lambda)|$], it follows that the maximal (minimal) extensions of the sets $U_\eta(\omega)$, $1 \leqslant \eta < \hat{\eta}$, are along the star lines of S_β^+ (S_β^-) and S_β^- (S_β^+) in the cases $\omega > 1$ ($\omega < 1$ respectively). Thus, as already indicated in the introduction, overrelaxation in the nonnegative and underrelaxation in the nonpositive case are appropriate.

(2) By the same considerations, we can also treat the case of complex ω. The resulting curves $q_\omega(\partial D_\eta)$ again turn out to be hypocycloids, now rotated by an angle of magnitude $\psi = (1/p)\arg(\omega - 1) - \arg \omega$. By easy geometric arguments, it is possible to show that complex ω do not yield better convergence results in the cases treated in this paper. However, if the eigenvalues of J_A are contained in sets S_β^+ rotated by a certain angle, special complex ω are appropriate. (See Kredell [7] for the case $p = 2$; for further results on complex ω, see Varga and Buoni [18], who consider the best relaxation factor $\omega = re^{i\theta}$, r small, over θ real. For each r sufficiently small, it is shown that there is a unique $\theta^*(r)$ which gives the optimum $\omega = re^{i\theta}$ on that circle.)

(3) In this paper, we have studied the convergence properties of the SOR iterative method when applied to a linear system the associated Jacobi matrix of which is weakly cyclic of index p and consistently ordered. However, we can also treat the more general inconsistent case (cf. Varga [17, Exercise 2, p.

109]), where J_A has the normal form

$$\begin{bmatrix} 0 & \cdots & 0 & B_{1,k+1} & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & & & & & & B_{p-k,p} \\ B_{p-k+1,1} & 0 & \cdot & \cdot & \cdot & 0 & \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & B_{p,k} & 0 & \cdots & 0 \end{bmatrix}$$

(see also Nichols and Fox [8]). Instead of the p-step relaxation method (2.2), we now consider the iteration

$$\mathbf{x}^{(m)} = \omega J_A \mathbf{x}^{(m-p+k)} + (1-\omega)\mathbf{x}^{(m-p)} + \omega \mathbf{c}. \qquad (*)$$

Again it is easy to show that the SOR method—if applied to $\mathbf{x} = J_A\mathbf{x} + \mathbf{c}$—converges iff $(*)$ does, and in case of convergence, it is p times faster than $(*)$. The study of convergence properties now leads us to the mapping

$$q_\omega(\lambda) = \frac{1-(1-\omega)\lambda^p}{\omega\lambda^{p-k}},$$

and again, the images $q_\omega(D_\eta)$, $\eta \geqslant 1$, are essential for questions of convergence.

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